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Int. J. Heat Mass Transfer. Vol. 13, pp. 1368–1371. Pergamon Press 1970. Printed in Great Britain

DISPLACEMENT THICKNESS OF AN UNSTEADY BOUNDARY LAYER WITH SURFACE MASS TRANSFER

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(*Received* 29 *September* 1969 *and in revised form* 8 *January* 1970)

NOMENCLATURE

a_∞,	ambient sound speed;
F,	function defining the displacement surface;
h,	a normal distance from the plate, slightly larger than the boundary-layer thickness;
$M_\infty(t)$,	instantaneous plate Mach number, $U_\infty(t)/a_\infty$;
$\dot{m}(x, t)$,	surface mass flux normal to the plate, equal to $\rho_w v_w$;
$Re(x, t)$,	Reynolds number, $xU_\infty(t)/\nu$;
t,	time;
(x, y),	Cartesian coordinate system, fixed relative to the plate, see Fig. 1;
(u, v),	velocity vector of the boundary layer flow field;
$U_\infty(t)$,	plate speed;
α,	a constant of order unity;
$\delta^*(x, t)$,	a quantity defined by equation (5.1);
$\delta_\rho(x, t)$,	a quantity defined by equation (5.2);
$\Delta^*(x, t)$,	displacement thickness for unsteady flows with surface mass transfer;
ν,	"scaled" kinematic viscosity, being a constant equal to $C\nu_\infty$;
$\rho(x, t)$,	density.

Subscripts

e (or ∞),	conditions at the outer edge of the boundary layer;
w,	conditions at the surface uf the plate;
1, 2,	conditions at $x = x_1$ and $x = x_2$, respectively.

1. INTRODUCTION

The concept of the displacement thickness of a viscous boundary layer is very useful and important, particularly in studying the viscous-inviscid interaction effects [1]. For steady flows with no surface mass transfer, the procedure for calculating this thickness is standard and straightforward (see e.g. Schlichting [2]). When the boundary layer is unsteady, the displacement surface can also be found by regarding such a surface as a fictitious solid boundary (impermeable) placed in the given free stream, and the unsteady, inviscid boundary condition on such a boundary leads to a normal velocity distribution just the same as that given by the boundary layer solutions at the outer edge. This was first done by Moore and Ostrach [3] who derived a differential equation for such a surface, valid for general, unsteady boundary layers, but without surface mass transfer.

With surface mass flux, the effective displacement thickness of a boundary layer has been studied by Mann [4] for the simple geometry of a flat plate in parallel motion. The analysis was later generalized by Hayasi [5] to account for arbitrary geometries. However these analyses were all aimed at steady flow situations.

In many practical applications, such as flights of rockets, missiles or re-entry vehicles, a continuously varying flight speed is often encountered. It is therefore of importance to

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study the unsteady boundary layers, and sometimes, to investigate the effective means for their cooling and controls. This in turn warrants the need for studying the unsteady boundary-layer flows including the effect of surface mass transfer. This study has recently been done for the special case of a semi-infinite flat plate in nearly quasi-steady motion parallel to itself with a particular form of surface mass flux distribution [6].

The purpose of this note is to show how the previous studies on the displacement thickness of a boundary layer can be extended and generalized to situations involving both flow unsteadiness and surface mass transfer. Again the simple geometry of a flat plate moving in its own plane is chosen to demonstrate the main idea, the surface mass flux being allowed to have arbitrary distributions, $\dot{m}(x, t)$. This study evidently has the significance of being a necessary first step toward understanding the various consequences due to the displacement effect of such boundary layers, including the induced pressures.

It is noted here that the combined effects of surface mass transfer and flow unsteadiness on the displacement thickness of a boundary layer, even for this simple configuration, do not seem to have been discussed before.

2. ANALYSIS

Consider now the problem of a sharp-edged semi-infinite flat plate moving with an arbitrary, time-dependent velocity, $U_\infty(t)$, into a compressible fluid at rest, an arbitrary form of $\dot{m}(x, t)$ being prescribed on the surface of the plate. It is convenient to study the problem with a coordinate system (x, y) fixed with respect to the plate (see Fig. 1). The transformation of the equations of motion from an inertial frame of reference to this generally non-inertial frame gives rise to an additional term in the x-momentum equation, well known as the "apparent" pressure gradient term.

In the present note, we will assume that the solutions to the unsteady, boundary-layer flow are known in this plate-fixed coordinate system, and propose to study the corresponding displacement thickness.

Let the unsteady displacement surface be represented by the equation

$$F(x, y, t) \equiv y - \Delta^*(x, t) = 0, \tag{1}$$

where, obviously, $\Delta^*(x, t)$ is the displacement thickness to be determined.

As is implied in the definition of the effective displacement surface, $F(x, y, t) = 0$ is to represent a fictitious solid boundary placed in an unsteady, inviscid free stream with velocity $U_\infty(t)$. On this fictitious surface, a vertical velocity distribution, $v_e(x, t)$, is presumed known from the solutions of the appropriate unsteady boundary layer flow.

It is then not difficult to obtain a relationship between $v_e(x, t)$ and $\Delta^*(x, t)$ through the boundary condition of inviscid flow over an impermeable solid surface, namely, that a fluid particle can only slide along the surface $F = 0$. The approximate expression is (see [7] for details)

$$v_e(x, t) \simeq U_\infty(t)\frac{\partial \Delta^*}{\partial x} + \frac{\partial \Delta^*}{\partial t}. \tag{2}$$

The approximation involved in the above result is related to assuming that v_e can be taken to be equal to the component of the fluid velocity normal to the surface $F = 0$ when viewed from a frame of reference in which the plate is moving into fluid at rest; the error being of order $(\partial \Delta^*/\partial x)^2$, usually very small.

Next we will attempt to transform equation (2) into a somewhat more convenient form involving familiar quantities in the boundary layer theory. This can be accomplished by constructing a control volume and considering its mass conservation.

Consider a control volume bounded by four plane boundaries, namely, two vertical planes: $x = x_1$, $x = x_2$ and two horizontal planes: the plate itself and $y = h$ where h is chosen to be slightly larger than the boundary-layer thickness. (See Fig. 2.)

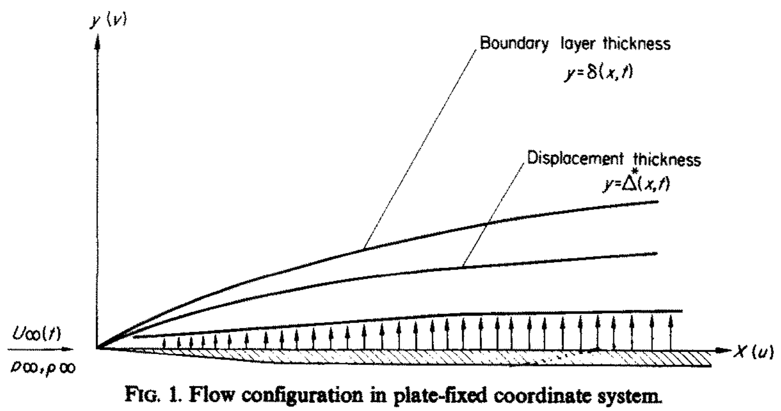

FIG. 1. Flow configuration in plate-fixed coordinate system.

A mass balance immediately yields the following integral relation:

$$\int_0^h (\rho u)_1 \, dy + \int_{x_1}^{x_2} (\rho_w v_w - \rho_e v_e) \, dx - \int_0^h (\rho u)_2 \, dy = \frac{\partial}{\partial t} \int_0^h \int_{x_1}^{x_2} \rho \, dx \, dy. \quad (3)$$

We then let x_2 approach x_1 arbitrarily, i.e. $x_2 = x_1 + \Delta x$. Assuming the boundary-layer solutions and surface mass flux are analytic and possess Taylor series expansions, we obtain immediately the following relation after applying the limiting process of $\Delta x \to 0$ to equation (3):

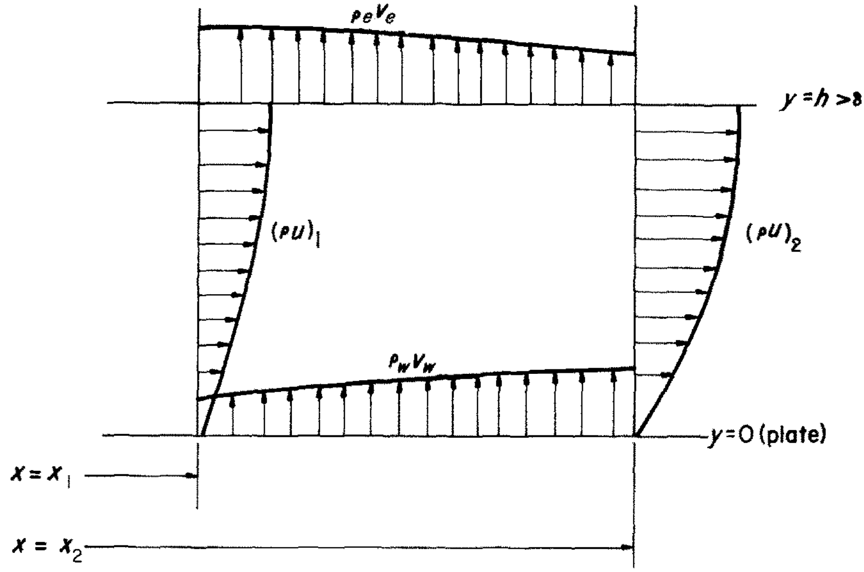

FIG. 2. Control volume for mass balance.

$$\rho_w v_w - \rho_e v_e - \frac{\partial}{\partial x} \int_0^h \rho u \, dy = \frac{\partial}{\partial t} \int_0^h \rho \, dy. \quad (4)$$

Since in the problem considered, $\rho_e = \rho_\infty$ = ambient density = constant and $U_\infty = U_\infty(t)$, we can write equation (4) in the following form:

$$\rho_w v_w - \rho_\infty v_e + \rho_\infty U_\infty \frac{\partial}{\partial x} \delta^* + \rho_\infty \frac{\partial}{\partial t} \delta_\rho = 0, \quad (5)$$

with

$$\delta^*(x, t) \equiv \int_0^\infty \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right) dy, \quad (5.1)$$

$$\delta_\rho(x, t) = \int_0^\infty \left(1 - \frac{\rho}{\rho_\infty}\right) dy. \quad (5.2)$$

Finally we obtain the desired differential equation for the determination of the quantity $\Delta^*(x, t)$ by combining equations (5) and (2). It is

$$\frac{\partial \Delta^*}{\partial x} + \frac{1}{U_\infty} \frac{\partial \Delta^*}{\partial t} = \frac{\partial}{\partial x} \delta^* + \frac{1}{U_\infty} \frac{\partial}{\partial t} \delta_\rho + \frac{\rho_w v_w}{\rho_\infty U_\infty}. \quad (6)$$

Equation (6) determines $\Delta^*(x, t)$ in terms of the known quantities δ^*, δ_ρ and $\rho_w v_w / \rho_\infty U_\infty$.

3. DISCUSSION AND CONCLUDING REMARKS

It is noted that equation (6) reduces to that obtained by Mann [4] when steady boundary layers are considered. On the other hand, it checks with Moore and Ostrach's [3] results for this particular geometry of a flat plate moving in its own plane, when $\rho_w v_w$ vanishes.

It should be noted that the left hand side of equation (6), as a whole, represents the instantaneous deflection angle of the streamline at the outer edge of the unsteady boundary layer, and is directly related to the calculation of unsteady weak interaction pressures [7]. In many practical applications, induced pressure calculation in particular, the form of the displacement thickness, $\Delta^*(x, t)$, itself seems to be of less importance than its effect on the deflection of the outer inviscid flow. Therefore, equation (6) can be used directly in many circumstances without having to be solved explicitly for $\Delta^*(x, t)$.

To be pointed out also is the fact that, while it might appear to be a simple extension of the result due to Moore–Ostrach [3], the result obtained here, equation (6), is by no means trivial. The explicit, additional term $\rho_w v_w / \rho_\infty U_\infty$, in equation (6) does not entirely represent the contribution of the surface mass flux to the displacement effect. An implicit contribution also enters through the quantities δ^* and δ_p, and these quantities are based on boundary-layer profiles which have also, implicitly, included the effect of blowing (or suction) through the appropriate boundary conditions used in solving the boundary-layer equations.

For flows with moderate Mach numbers $[M_\infty^2 \simeq 0(1)]$, the additional term $\rho_w v_w/\rho_\infty U_\infty$, may have significant contributions to the displacement thickness, because its magnitude may well be comparable to that of the term $\partial\delta^*/\partial x$.

For nearly quasi-steady hypersonic flows with "similarity type" of blowing (or suction), i.e. $\rho_w v_w/\rho_\infty U_\infty = \alpha/(Re)^{\frac{1}{2}}$, as treated in [6], this additional term is found to be small compared to $\partial\delta^*/\partial x$; however, it may become comparable to the term $U_\infty^{-1}(\partial\delta_p/\partial t)$, depending on the parameters characterizing the flow unsteadiness. A discussion on boundary layer induced pressures for such flows also appears in [6].

Finally we remark that the idea used in this note can be employed to generalize the study to more general flow configurations. Thus in two dimensional flow, for example, we shall have the situations of $U_\infty = U_\infty(x, t)$, $\rho_\infty = \rho_\infty(x, t)$ for an arbitrary body in unsteady motion.

ACKNOWLEDGEMENTS

This work was supported by the Air Force Office of Scientific Research (AFOSR) under Grant AF-AFOSR-68-1581. The author acknowledges some helpful comments of E. Reshotko.

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Int. J. Heat Mass Transfer. Vol. 13, pp. 1371–1375. Pergamon Press 1970. Printed in Great Britain

EXTENSION OF THE NUMERICAL METHOD FOR MELTING AND FREEZING PROBLEMS

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(*Received* 19 *January* 1970)

NOMENCLATURE

- D, horizontal dimension of test cell [cm];
- E, vertical dimension of test cell [cm];
- k, thermal conductivity [cal/cms °C];
- k_{eff}, effective thermal conductivity [cal/cms °C];
- L, vertical dimension of liquid [cm];
- n, nodal position in space network;
- N, total number of space nodes;
- Nu, Nusselt number, k_{eff}/k_2 [dimensionless];
- Q, heat flux [cal/cm^2 s];
- r, number of spatial node at solid–liquid interface;
- t, time [s];
- T, temperature [°C];
- T_F, fusion temperature [°C];
- u, velocity [cm/s];
- x, distance along vertical axis [cm].

Greek symbols

- α, thermal diffusivity [cm^2/s];
- ϵ, ice thickness, interface position [cm];
- θ, point temperature minus fusion temperature, $T - T_F$ [°C];
- λ, latent heat of fusion [cal/g];
- ρ, density [g/cm^3].

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